INVERSE BOUNDARY-VALUE PROBLEMS OF HEAT CONDUCTION

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Possible formulations of the problems of determining heat fluxes and temperatures at the boundary of a solid from known temperatures within the solid are examined. A classification of these formulations is offered. Various methods for solving one-dimensional inverse problems are analyzed.

A characteristic of heat transfer in a solid is a significant smoothing of features in the boundary functions with distance from the heat-exchange surface into the object. The rate of change of the temperature at a point deep in the interior can turn out to be far lower than the rate of change of the temperature at the external surface. This physical nature of heat propagation leads to a familiar pathological singularity in inverse problems: The results are not continuous functions of the input temperature data (the Hadamard conditions of correctness are violated [1-4, 37]). Since inverse heat-conduction problems usually involve the processing and interpretation of the results of real thermal experiments, there are errors in the input data. In an exact solution of the problem (provided, of course, that an exact solution is possible), the errors in the input data can be considerably amplified. For this reason, the solution of inverse heat-conduction problems should be based on those approximate methods which are capable of suppressing instabilities in the results while providing the desired accuracy. Our purpose in the present paper is to briefly review and systematically classify certain methods for solving inverse boundary-value problems.

We consider a quite general formulation of the one-dimensional inverse problem. We are to determine the boundary conditions and temperature field in an object in which the heat transfer is described by a generalized quasilinear heat-conduction equation with a given initial condition and with known temperatures at two points within the object. These points and the boundaries of the object are movable. Their motion is described by known functions. We thus have

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)+k(T)\frac{\partial T}{\partial x}+\psi(T), \tag{1}$$

$$0<\tau\leqslant\tau_m,\quad X_1(\tau)<x<X_4(\tau),$$

$$T(x,\ 0)=\varphi(x), \tag{2}$$

$$T(X_2(\tau),\ \tau)=f_2(\tau),\quad T(X_3(\tau),\ \tau)=f_3(\tau), \tag{3}$$

$$\left.\begin{aligned}-\lambda\frac{\partial T(X_1(\tau),\ \tau)}{\partial x}&=q_1(\tau),\ T(X_1(\tau),\ \tau)=T_1(\tau),\\ -\lambda\frac{\partial T(X_4(\tau),\ \tau)}{\partial x}&=q_4(\tau),\ T(X_4(\tau),\ \tau)=T_4(\tau)\end{aligned}\right\} \tag{4}$$

where the two conditions at the right in (4) are the unknown conditions.

From the physical standpoint this formulation of the problem presupposes distributed heat sources in the object and the filtration through the object of a gaseous or liquid phase. The motion of the external

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boundaries can be caused by removal of mass, e.g., through ablation, while the motion of the points with the known temperatures can be governed by thermal shrinkage or extension of the material. This physical model is encountered in many experiments and incorporates a variety of particular cases.

The formulation in (1)-(4) of this problem is quite general, and to solve it we must put it in a more concrete form. Here we consider three possible cases.

I. Boundary-Value Formulation of the Inverse Heat-Conduction Problem

We assume that an algorithm for solving the corresponding direct problem, f = Au, exists and that we have found a method for "inverting" it in order to determine the unknown relationship (input data) → (boundary condtions) [u = R(f)].

1) For linear problems with movable boundaries such formulations can be obtained (and then solved) on the basis of the theory of thermal potentials. One particular example is the case in which there are no heat sources or filtration effects in the object, which has a constant thermal diffusivity; this approach was treated in [5, 6, 28].

2) Algorithms for solving linear inverse heat-conduction problems in the boundary-value formulation with movable boundaries can be considerably simplified by making use of the principle of expanding the initial region (x, τ) along the spatial coordinate to a rectangular region $\{X_{1min}(\tau) \le x \le X_{4max},\ 0 \le \tau \le \tau_m\}$. Then the new temperature data at the fixed points in the object, with coordinates X_{2max} and X_{3max}, obtained from a solution of the direct heat-conduction problem in the region $\{X_2(\tau) \le x \le X_3(\tau),\ 0 \le \tau \le \tau_m\}$, are incorporated. As a result, the original inverse heat-conduction problem in the boundary-value formulation can be divided into two inverse problems involving determination of fictitious temperatures or heat fluxes at new boundaries introduced in accordance with some convention. These problems are solved (e.g., through the use of the integral Duhamel form) in the regions

$$\{X_{1_{\min}}(\tau) \leqslant x \leqslant X_{3_{\min}},\ 0 \leqslant \tau \leqslant \tau_m\} \text{ and } \{X_{2_{\max}} \leqslant x \leqslant X_{4_{\max}},\ 0 \leqslant \tau \leqslant \tau_m\},$$

respectively [5]. Finally, the unknown conditions can be found by solving the corresponding direct heat-conduction problems, since the real boundaries of the object are incorporated in the new regions.

This method of fictitious boundaries suffers from the disadvantage that the accuracy and stability of the new inverse problems are worse than those of the original problem, because the points of the fictitious boundaries are far from the points with the input temperatures.

3) Inverse problems in the boundary-value formulation can be solved by numerical methods through the use of a variety of difference schemes (explicit and implicit). In this case the heat-conduction equation is integrated along the direction of the time variable [7]. For inverse heat-conduction problems with movable boundaries, it turns out to be advantageous to first transform the original regions $\{X_1(\tau) \le x \le X_2(\tau),\ 0 \le \tau \le \tau_m\}$, $\{X_2(\tau) \le x \le X_3(\tau),\ 0 \le \tau \le \tau_m\}$, $\{X_3(\tau) \le x \le X_4(\tau),\ 0 \le \tau \le \tau_m\}$ into corresponding rectangular regions, by introducing new spatial variables of the type [8, 9]

$$\xi_j = \frac{x - X_j(\tau)}{X_{j+1}(\tau) - X(\tau)_j},\quad j = 1,\ 2,\ 3.$$

II. Inverse Problems in the Cauchy Formulation

We seek a continuation of the solution of the heat-conduction equation from the boundary of the small region in which the temperature and heat flux are given (Cauchy data) to a larger region, out to the boundaries of this region with the unknown conditions. A general characteristic of this formulation of the inverse heat-conduction problem is the need to carry out a preliminary calculation of the heat flux at the lines $X_2(\tau)$ and $X_3(\tau)$ in the solution of the direct problem in the region $\{X_2(\tau) \le x \le X_3(\tau),\ 0 \le \tau \le \tau_m\}$.

1) In this case we can also use the method of thermal potentials for linear problems. The continuation of the temperature field for a linear heat-conduction equation can be written as certain infinite series in terms of arbitrary input functions [10, 11] (under the assumption that these functions are differentiable an unlimited number of times).

2) Many algorithms for inverse problems in this formulation, including the original nonlinear case, (1)-(4), can be found by difference methods through the use of explicit and implicit approximation schemes

[7,12]. In this case the heat-conduction equation is integrated along the direction of the spatial variable, toward the boundary with the unknown solution.

III. Variational Forms of the Inverse Heat-Conduction Problem

This case covers a broad class of possible formulations of inverse problems associated with the seeking of extrema of corresponding functionals. There are two possible cases.

1) We are given a variational formulation of the heat-conduction equation, and we seek a solution of the problem which leads to a steady-state functional, which is the basis of the variational principle. We note that, despite the present lack of methods for solving the inverse heat-conduction problem on the basis of variation principles, this approach holds much promise.

2) The inverse heat-conduction problem specified by the system of differential equations in (1)-(4) is interpreted as an optimum-control problem. We are to choose the optimum control u (the temperature or the heat flux at the boundary of the object) such that a target functional is minimized; the role of this functional is played by a discrepancy taken in the norm of the space F which is chosen (this space is usually Euclidean or L_2):

$$\|Au - f\|_F^2 - \min_u .$$

To solve the inverse heat-conduction problem in the variational formulation we can use, in particular, the method of least squares [13-16], gradient-type search methods [17,18], or a trial-and-error method [20].

Since the original formulation of the inverse boundary-value problem is incorrect in the classical sense, the various methods used for direct solutions of this problem (unless this incorrectness is taken into account) turn out to be potentially unstable. Here we refer to these methods as "direct methods."

In practice, the use of direct methods rests on natural regularizing properties which some method or computational logarithm may have, to some degree or other. The reason for the natural regularization of the solutions of the inverse heat-conduction problem lies in the physics of heat propagation in an object, which results in a regularization of the heating regime [19] at a point in the object at which a temperature pickup is placed. It is primarily this effect which governs the principle for choosing the time intervals for calculations from the condition for the suppression of an undesirable "buildup" of the results when direct algebraic methods are used for solving the integral forms of inverse problems [13,21-31]. This principle was used in its most explicit form in the initial version of the method of sequential intervals [21], in which the calculation step used in the determination of step heat-flux functions is chosen such that the temperature within an infinite plate varies linearly within this time interval. To a certain extent, thermal regularization is also of assistance in combating instabilities when trial-and-error or least-squares methods are used [13-16] or in various difference methods for solving the inverse heat-conduction problem [7,12].

Furthermore, a natural regularization of the solutions of the inverse problems can be related to the "viscous" properties of the calculation schemes based on some algorithm or other. This assertion applies primarily to numerical methods for solving the inverse heat-conduction problem [7,12,14].

Accordingly, for each such algorithm based on direct methods we can specify a critical calculation step (over which the increment in the Fourier number is chosen on the basis of the considerations of thermal similarity theory) which provides a sufficiently regular behavior of the unknown solutions. The critical value ΔFo_{cr} and the accuracy of the solution of the inverse problem depend on errors in the input data, so that when direct methods are used particular attention must be paid to the original processing (preparation) of the raw data.

The critical value ΔFo_{cr} is also governed by the a priori specified order of the approximation of the unknown function, e.g., in various algebraic schemes of the integral forms of the inverse heat-conduction problem or in the method of least squares. An effort to improve the accuracy of the approximation, i.e., to discern subtler features in the unknown function, by improving the approximation inevitably leads to an increase in ΔFo_{cr}, so that the solution may not be improved.

Among the most commonly used methods for processing the data of thermal experiments are those based on an algebraic solution of the integral forms of the inverse heat-conduction problem in the boundary-value formulation. A large number of difference methods have been developed for solving essentially the same inverse problems. There is accordingly a need for a comparative analysis of these algorithms, especially since most of them are constructed on the basis of roughly the same principles for approximating the integral equations.

Let us examine the problem of determining the boundary conditions for one-dimensional objects with fixed boundaries and constant thermal properties.

In this case the general integral form of the inverse heat-conduction problem is

$$Au \equiv \int_0^{Fo} u(\xi)\, k(Fo, \xi)\, d\xi = f(Fo), \quad 0 < \xi \leqslant Fo_m, \tag{5}$$

where u(Fo) is the unknown solution (the temperature, heat flux, or auxiliary function).

For steps $\Delta Fo = (Fo_m/m)$ of fixed magnitude we find, after the approximation of (5), a system of linear algebraic equations with a lower triangular matrix having the property that the elements along a diagonal are equal:

$$A_\Delta \hat{u} \equiv \sum_{i=1}^{n} \varphi_{i,n} \hat{u}_i = f_n, \quad n = 1, 2, \ldots, m. \tag{6}$$

Here $\varphi_{i,n}$ and $\hat{u}_i$ can be determined from, e.g.,

$$\varphi_{i,n} = \int_{\tau_{i-1}}^{\tau_i} K(Fo, \xi)\, d\xi, \quad \hat{u}_i = \frac{u_i + u_{i-1}}{2}.$$

We introduce the "spectral conditionality number" (the product of the spectral norm of the matrix A_Δ and the spectral norm of its inverse matrix [32-34]):

$$C(A_\Delta) = \|A_\Delta\|_2 \cdot \|A_\Delta^{-1}\|_2.$$

The norms of matrices A_Δ and A_Δ^{-1} are governed by the largest and smallest eigenvalues of the corresponding normal matrix,

$$\|A_\Delta\| = \max \lambda_{A_\Delta^T A_\Delta}, \quad \|A_\Delta^{-1}\| = \min \lambda^{-1}_{A_\Delta^T A_\Delta}$$

and are found in an iterative manner from the condition for the steady state of the following functional:

$$\|A_\Delta\| = \max_{\hat{u}} \frac{(A_\Delta^T A_\Delta \hat{u}, \hat{u})}{(\hat{u}, \hat{u})}, \quad \|A_\Delta^{-1}\| = \min_{\hat{u}} \frac{(A_\Delta^T A_\Delta \hat{u}, \hat{u})}{(\hat{u}, \hat{u})}.$$

The conditionality number $C(A_\Delta)$ sets an upper limit on the ratio of the relative mean square error of the solution of the system to the relative mean square error of the vector on the right side:

$$\frac{\|\delta u\|}{\|\hat{u}\|} \leqslant C(A_\Delta) \frac{\|\delta f\|}{\|f_0\|},$$

where $\hat{u}_0$ and f_0 are the "exact" vectors.

For the case in which the input information f is specified exactly, but the matrix elements are perturbed, we can write [33]

$$\frac{\|\delta \hat{u}\|}{\|\hat{u}_0 + \delta \hat{u}\|} \leqslant C(A_\Delta) \frac{\|\delta A_\Delta\|}{\|A_\Delta\|}.$$

Hence the deviation $\delta\hat{u}$ divided by $\hat{u}_0 + \delta\hat{u}$, is bounded by the relative "error" of the matrix A_Δ multiplied by the conditionality number.

The quantity $C(A_\Delta)$ can thus be interpreted as a measure of the quality of matrix A_Δ for a comparative analysis of various algorithms for various values of ΔFo: The higher $C(A_\Delta)$, the less stable the inverse matrix A_Δ^{-1}.

In those cases in which the basic result of the analysis is to be a choice of some algorithm or other, a criterion for the comparison can be the product of the euclidean norms $N = \left(\sum_{i,n} a_{i,n}^2\right)^{1/2}$ of the matrices

A_Δ and A_Δ^{-1}, since the conditionality number satisfies the inequalities

$$\frac{1}{m}\gamma(A_\Delta) \leqslant C(A_\Delta) \leqslant \gamma(A_\Delta),$$

where $\gamma(A_\Delta) = N(A_\Delta)N(A_\Delta^{-1})$.

As an example, Fig. 1 shows a plot of $\gamma(A_\Delta \Delta Fo)$ for the cases corresponding to Table 1. When direct methods are used to solve the inverse heat-conduction problem in the boundary-value formulation and in the Cauchy formulation, one of the basic questions is that of determining the critical increment of the Fourier number. We know that an increase in the magnitude of the coefficients of the matrix A_Δ along the main diagonal leads to an improvement of the conditionality of the matrix. This improvement can be achieved by increasing the time step (in Fo), since the kernels of the corresponding integral equations, $K(\tau - \xi)$, have maxima $K_{max} = K(\tau - \xi^*)$ [as $\xi \to \tau$, we have $K(\tau - \xi) \to 0$]. The physical meaning of this result is that when there is a unit boundary condition at one surface and a zero boundary condition at the other, the rate of increase of the temperature at a given point begins to decrease after the time $\xi = \xi^*$. An exceptional case is that of a plate which is thermally insulated on one side; in this case the known temperature at this side of the plate (or in some vicinity of this side) is used to determine the heat flux to the other side. In this case the kernel of the integral equation is a monotonically increasing function, which gradually converts into a regular-heating regime.

For other inverse problems we can specify a number $\Delta\overline{Fo}$ such that if $\Delta Fo > \Delta\overline{Fo}$ the elements on the diagonal of A_Δ turn out to be predominant. In particular, if we use a stepped approximation of the boundary condition of the first kind with equal intervals, this situation occurs for a semi-infinite object if $\Delta Fo > \sim 1$, while for a plate it occurs if $\Delta Fo > \sim 0.8$. Here $\varphi_{1n} < \varphi_{2n} < \ldots < \varphi_{nn}$ ($n > i$, $n = 1, 2, \ldots, m$). This case corresponds to the "limiting" natural thermal regularization of the problem.

We note that if the temperature pickup is placed at the heat-exchange surface at which the boundary condition is to be found, the maximum values of the coefficients of the matrix A_Δ always lie on the diagonal for any arbitrarily small values of ΔFo.

The values of ΔFo obtained in this manner turn out to be too large in many problems, and this approach for choosing the step can be recommended only for relatively slow thermal processes which occur over long time intervals.

As the calculated simulations show, the value of ΔFo can be reduced considerably in comparison with that in the case under consideration, until the results of the solution of the inverse heat-conduction problem become significantly unstable. If the following approach is taken, the critical values of these steps can be found without "experimentally grouping around for" the instability threshold in the solution of the methodological examples.

We require that for a stepped approximation of the unknown function the values of ΔFo satisfy the condition

$$\Delta Fo > \Delta Fo_{cr} = Fo^*, \ \Delta Fo \simeq \Delta Fo_{cr},$$

where $Fo^* = [a(\tau - \xi^*)/l^2]$ is the Fourier number corresponding to the maximum of $K(\tau - \xi)$.

Then when exact (or approximately exact) input information is used, the solution of Eq. (6) becomes quite smooth.

Table 1 and Fig. 2 show estimates of the critical steps obtained in this manner for several models and for several methods for solving the inverse heat-conduction problem. Analysis of the stability boundaries of the inverse heat-conduction problem shows that under otherwise equal conditions the problem of reconstructing the temperature of the boundary of the object by a direct method is always stabler than the analogous problem for the heat flux.

This approach to the choice of the calculation steps ΔFo can also be extended to inverse problems with moving boundaries.

For any algorithm for solving the inverse problem which is regularized in terms of the calculation step, the value of ΔFo can be estimated when there are errors in the input data from the condition that the discrepancy match the magnitude of the errors. For example, the choice of the number of intervals m can be made from the condition

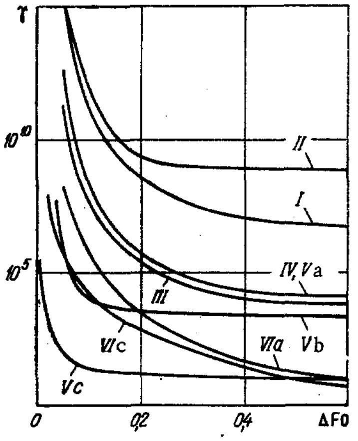

Fig. 1

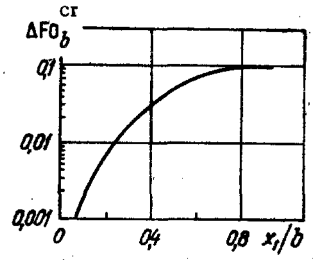

Fig. 2

Fig. 1. Conditionality numbers for various models of the solution of linear inverse problems for a stepped approximation of the unknown function (m = 50). The curves are labelled with the numbers of the models listed in Table 1. I, II, III, IVa) $x_1 = b$; Vb, VIb) $x_1 = 0.5$ b; Vc) $x_1 = 0.05$ b; VIa) $x_1 = 0.95$ b.

Fig. 2. Critical values of the steps for model VI of the inverse problem.

$$\sum_{n=1}^{m}\left[\frac{T_n(m)-f_n}{\bar{\sigma}_n}\right]^2 \simeq m,$$

where $\bar{\sigma}_n$ is the estimate of the mean square error of the input data for time n.

We note that the values of ΔFo_{cr} fall off rapidly as the internal point of the object with the specified temperature approaches the heat-exchange surface. With identical steps ΔFo we have the best conditionality for the problem of determining the heat flux at the boundary of an object from temperature measurements at this boundary (the problem of transforming boundary conditions) [37]. Accordingly, in the experiments the temperature pickups should be moved as close as possible to the boundary of the object where the boundary condition is to be determined.

When methods based on the method of least squares or self-similar solutions are used, the role of the natural regularization parameter can be played by the degree of the approximating polynomial for the boundary condition being sought for. The "best order" of the approximation should be determined from the condition

$$\int_0^{\tau_m}[T(u^j, \tau)-f(\tau)]^2\,d\tau \simeq \delta_{L_2}^2,$$

where u^j is the approximation of "j-th order" of the unknown boundary condition, and δ_{L_2} is the error of the input data in

$$L_2\left(\delta_{L_2}^2=\int_0^{Fo_m}\sigma^2(Fo)\,dFo\right).$$

To solve inverse problems of optimum control, iterative methods can be used, in particular, the methods of steepest descent and of conjugate gradients [18]. Whether these methods can be used successfully depends on whether the convergence rate decreases rapidly as successive iterations are carried out; this decrease makes it possible to avoid the danger that unstable results will appear.

The target functional which determines the degree of deviation of the "experimental" temperature $f(\tau)$ from the calculated temperature $T(u, \tau)$ for a given equation u, can be associated with the quadratic measure of the error:

$$J(u)=\int_0^{\tau_m}[f(\tau)-T(u, \tau)]^2\,d\tau. \tag{7}$$

The gradient of the functional, $J' = (\partial J/\partial u)$, can be calculated either through an immediate differentiation of (7) or through a solution of the auxiliary boundary-value problem which is the adjoint of the

TABLE 1. Certain Linear Inverse Problems

No	Model	Reference	ΔFo^{cr}
I	Semi-infinite object $q — ?$ at 0; point x_1; $\longrightarrow x \to \infty$ Plate (function g) $q_1 — ?$ at 0; $q_2 = 0$ at $x_1 = b$; $\longrightarrow x$ $\varphi_{in} = \dfrac{2x_1 \sqrt{\Delta Fo_{x_1}}}{\lambda_0} \times (n-p) i\Phi^* \left[\dfrac{1}{2\sqrt{\Delta Fo_{x_1}(n-p)}} \right]_{p=i}^{p=i-1}$	[19]—Eq.(19), Ch.5 [22]—Eq. (8), (9) [6]—Eq. (6), (14)	$\Delta Fo_{x_1}^{cr} \approx 0{,}4 \div 0{,}5$
II	Plate $q_1 — ?$ at 0; point x_1; $q_2 = 0$; $\longrightarrow x$ $\varphi_{in} = \dfrac{2b\sqrt{\Delta Fo_b(n-p)}}{\lambda_0} \sum_{j=0}^{\infty} \left\{ i\Phi^* \times \left[\dfrac{2j + \frac{x_1}{b}}{2\sqrt{\Delta Fo_b(n-p)}} \right] + i\Phi^* \left[\dfrac{2(j+1) - \frac{x_1}{b}}{2\sqrt{\Delta Fo_b(n-p)}} \right] \right\}_{p=i-1}^{p=i}$	[19]—Eq. (24), Ch. 5 [22]—Eq. (21), (22) [6]—Eq. (4), (14)	$\Delta Fo_b^{cr} \approx 0{,}5 \left(\dfrac{x_1}{b} \right)^2$
III	Plate $T — ?$ at 0; point x_1; $q = 0$ at b; $\longrightarrow x$ $\varphi_{in} = \sum_{j=1}^{\infty} (-1)^{j+1} \left\{ \Phi^* \left[\dfrac{(2j-1) - \frac{x_1}{b}}{2\sqrt{\Delta Fo_b(n-p)}} \right] + \Phi^* \left[\dfrac{(2j-1) + \frac{x_1}{b}}{2\sqrt{\Delta Fo_b(n-p)}} \right] \right\}_{p=i-1}^{p=i}$	[19]—Eq. (28), Ch. 4	$\Delta Fo_b^{cr} \approx 0{,}17 \left(\dfrac{x_1}{b} \right)^2$
IV	Semi-infinite object $T — ?$ at 0; point x_1; $\longrightarrow x \to \infty$ $\varphi_{in} = \Phi \left[\dfrac{1}{2\sqrt{\Delta Fo_{x_1}(n-p)}} \right]_{p=i}^{p=i-1}$	[19]—Eq. (6), Ch. 4	$\Delta Fo_{x_1}^{cr} = 0{,}17$
V	Plate (function g) $T — ?$ at 0; $q = 0$ at b; $\longrightarrow x$ $\varphi_{in} = \Phi^* \left[\dfrac{\frac{x_1}{b}}{2\sqrt{\Delta Fo_b(n-p)}} \right]_{p=i}^{p=i-1}$	[31]—Eq. (13)	$\Delta Fo_b^{cr} = 0{,}17 \left(\dfrac{x_1}{b} \right)^2$
VI	Plate $T — ?$ at 0; point x_1; $T_2 = 0$ at b; $\longrightarrow x$ $\varphi_{in} = \sum_{j=0}^{\infty} \left\{ \Phi^* \left[\dfrac{(2j+1) - \frac{x_1}{b}}{2\sqrt{\Delta Fo_b(n-p)}} \right] - \Phi^* \left[\dfrac{(2j+1) + \frac{x_1}{b}}{2\sqrt{\Delta Fo_b(n-p)}} \right] \right\}_{p=i-1}^{p=i}$	[19]—Eq. (47), Ch. 4 [50]—Eq. (7)	see Fig. 2

original problem (1), (2), (4) [18]. Experience in the use of these gradient methods for minimizing (7) has shown that, if the input data are sufficiently smooth, some 10–15 iterations, beginning with zeroth approximation, are required to obtain a good approximation of the desired boundary condition u. If there are errors in the input temperature, then the iterative process is ended in accordance with the discrepancy principle as soon as the condition

$$J(u^k) < \delta^2_{L_2}$$

is satisfied for the first time. In the last iteration, the depth of the step β_{k-1} is determined from the solution of the equation

$$J(u^{k-1} - \beta_{k-1} p^{k-1}) = \delta^2_{L_2}$$

where

$$p^{k-1} = J'^{k-1} + \gamma_{k-1} p^{k-2},$$

$$\gamma_{k-1} = \frac{\int_0^{\tau_m} [J'^{k-1}(\tau)]^2 \, d\tau}{\int_0^{\tau_m} [J'^{k-2}(\tau)]^2 \, d\tau} \quad (\gamma_0 = 0)$$

[in the method of steepest descent, all the γ_l (l = 0, 1, ...) vanish]. Accordingly, in the solution of the inverse heat-conduction problem (as an optimum-control problem) by means of gradient methods, the role of the natural regularization parameters is played by the number of the iteration and the size of the step β in the last iteration. Iterative methods are used successfully to solve inverse problems in linear and nonlinear formulations.

Accordingly, with a correct use of the natural regularization properties which are due to the heat-propagation process itself and which are incorporated in the calculation schemes, direct methods can be used successfully under certain conditions to solve the inverse heat-conduction problem, including problems in which there are errors in the input information.

There are many situations, however, in which direct methods will not be satisfactory. In the first place, they cannot be used to reconstruct structural details of the unknown boundary conditions if these conditions are changing rapidly (and it is precisely these problems which have become extremely important in a wide variety of experimental studies of heat transfer). Attempts to achieve good results under these conditions by raising the degree of the approximation of the boundary conditions, by increasing the number of iterations in the choice of the discrepancy being minimized in terms of the norm of the Sobolev space, W_2^n, of functions which are differentiable in a generalized manner (instead of L_2), do not have the desired effect, and if there are errors in the input information, the inevitable result is that the solution "gets out of hand."

Second, many thermal experiments are so brief that the critical values of the time steps turn out to be comparable to or even longer than the total duration of the experiment.

In all these cases, an effective solution of the inverse problems can be constructed by means of the general regularization method of Tikhonov [1, 2]. The corresponding algorithms for boundary-value formulations of linear and nonlinear problems were worked out in [4-6, 35, 36, 38].

Numerical solutions of nonlinear inverse heat-conduction problems in the Cauchy formulation can be regularized by introducing a system of regularizing functionals corresponding to spatial layers of the difference grid (with some boundary conditions specified at $\tau = \tau_m$)

$$\Phi[T_i, \alpha] = \|M_i T_i - g_i\|^2_{E_m} + \alpha_i \Omega_i, \quad i = 1, 2, \ldots, n, \tag{8}$$

where M_i is a three-diagonal matrix, governed by the implicit difference scheme chosen (in general, the coefficients depend on T_{in}), $T_i = [T_{i1}, T_{i2}, \ldots, T_{im}]^T$ is the unknown temperature vector in the i-th spatial layer, α_i is the regularization parameter, and Ω_i is the stabilizing functional. We can set

$$\Omega_i = k_1 \|\Delta T_i\|^2_{E_m} + k_2 \|\Delta^2 T_i\|^2_{E_m}, \quad k_1 \geqslant 0, \quad k_2 > 0,$$

which corresponds to a second-order regularization (Δ and Δ^2 are, respectively, the first and second differences on the time grid).

After the operations involved in minimizing (8) are carried out, we have a regularized system of nonlinear algebraic equations in each i-th coordinate cross section; from the solution of this system of equations for a given value of the parameter α_i we find $T_i^{\alpha_i}$.

The regularization parameters α_i can be chosen on the basis of the discrepancy principle of Morozov [41], with an automatization of the given process in accordance with [42]. According to simulations, in several cases it turns out to be possible to choose the regularized boundary function from some effective value of the parameter $\bar{\alpha}$, from the condition

$$\|T^{\bar{\alpha}} - f_\delta\|^2_{E_m} = \delta^2,$$

where $T^{\bar{\alpha}}$ is the temperature at the position of the temperature pickup, found from the regularized approximation of the boundary function for some value of the parameter $\bar{\alpha}$; also, δ is the discrepancy level, which is determined by the error of the input data f. The desired approximation in this problem can also be determined on the basis of the quasioptimum parameter of Tikhonov and Glasko [44], if one effective value is chosen for all the regularized systems. This method for regularizing the difference scheme for the solution of a nonlinear inverse heat-conduction problem in the Cauchy formulation is described in more detail in [39].

Iterative schemes for solving inverse problems can be regularized by a natural method which follows directly from the regularization method and which consists of adding an appropriate stabilizer to the initial target functional. The convergence of the iterative process in this case can be accelerated by combining generalized Newton methods [43] with the choice of the initial approximation by the conjugate-gradient method. The algorithms for solving inverse boundary-value problems discussed above frequently turn out to be effective when combined with each other. For example, the problem can be initially solved by some direct method, and the result can be adopted as the initial approximation (trial solution) in the regularized algorithm for a further refinement of this solution.

In summary, we note that the mathematical apparatus presently available for solving inverse heat-conduction problems has been worked out thoroughly enough that it can be used effectively in a variety of experimental problems [45, 46]. In particular, along with the conjugate heat-transfer problems [47, 48], the inverse boundary-value problems furnish an opportunity for studying complicated processes involving transient thermal interactions between solids and surrounding media [49].

NOTATION

A, operator; A_Δ, matrix approximating the integral operator; A_Δ^T, transposed matrix; b, plate thickness; C(T), specific heat at constant volume; $C(A_\Delta)$, "conditionality number"; f, input temperature data; g, auxiliary function (see [6, 31, 36]); m, number of steps in the time interval; q, heat flux; T_Δ, temperature; u, unknown solution of the integral equation; $\hat{u}$, solution of the algebraic system which approximates the integral equation; $X(\tau)$, coordinate of the moving boundary of the object or of the moving temperature pickup; x, running coordinate; x_1, coordinate of fixed temperature pickup; $\lambda(T)$, thermal conductivity; τ, running time; τ_m, final time; $\varphi(x)$, initial temperature distribution in the object; $\psi(T)$, distributed heat sources in the object; ΔFo, increment in the Fourier number; $\|\cdot\|$, norm.

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